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INITIAL-BOUNDARY PROBLEMS FOR SEMILINEAR HYPERBOLIC SYSTEMS WITH SINGULAR COEFFICIENTS

In the paper we use the framework of Colombeau algebras of generalized functions to study existence and uniqueness of global generalized solutions to mixed non-local problems for a semilinear hyperbolic system. Coefficients of the system as well as initial and boundary data are allowed to be strongly singular, as the Dirac delta function and derivatives thereof. To obtain the existence-uniqueness result we prove a criterion of invertibility in the full version of the Colombeau algebras.

1. Introduction.

In the domain $\Pi = \{(x,t)| -L < x < L,\ t > 0\}$ we consider the following initial-boundary value problem for a generalized function U:

$$(\partial_t + \Lambda(x,t)\partial_x)U = f(x,t,U), \qquad (x,t) \in \Pi, \tag{1}$$

$$U|_{t=0} = A(x), \qquad x \in (-L,L) \tag{2}$$

$$B(t)U|_{x=-L} + C(t)U|_{x=L} + \int_{-L}^{L} D(x,t)U\,dx = H(t), \qquad t \in (0,\infty) \tag{3}$$

where, U, f, A, and H are real n-vectors, Λ, B, C, and D are real $(n \times n)$-matrices, and $\Lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_n)$ is a diagonal matrix.

Special cases of (1)–(3) are mathematical formulations of problems arising in population dynamics [1, 7, 15], laser dynamics [5, 13, 14, 16], and chemical kinetics [17].

Our goal is to find global solutions to problem (1)–(3) when the data Λ, A, B, C, D, and H are allowed to be strongly singular (at least of the Dirac delta type). This entails multiplication of distributions in (1) and (3). Indeed, since initial singularities expand from $\partial\Pi$ into Π along characteristic curves of (1), one can expect that solutions within Π are at least as singular as they are on $\partial\Pi$. Furthermore, since the characteristics of (1) are singular themselves, we also meet the problem of composition of two singular functions (for instance, the composition of singular initial data with singular characteristic curves). It is known [3] that even if F is a regular function, but S is a singular one, then $F(S(x))$ is not well-defined in $\mathcal{D}'$. Finally, it should be noted that such three ingredients as singularities, nonlinear operations, and differentiation, cannot be presented unrestrictedly within $\mathcal{D}'$. All this makes impossible to use the framework of the distribution theory for our purpose. Nevertheless, such a differential-algebraic structure as an algebra of generalized functions is able to deal with the above problems in a quite reasonable way. We here use the Colombeau version $\mathcal{G}$ of an algebra, which is defined on any domain in $\mathbb{R}^n$ as well as on its closure, is a sheaf, and admits restrictions to the coordinate planes.

We hence assume that entries of A are generalized functions in the Colombeau algebra $\mathcal{G}[-L,L]$, entries of B, C, and H are from $\mathcal{G}(\mathbb{R}_+)$, and entries of Λ and D are from $\mathcal{G}(\overline{\Pi})$.

Another advantage of using Colombeau algebra of generalized functions lies in the fact that in a variety of important cases the division by generalized functions, in particular the division by discontinuous functions and measures, is defined in $\mathcal{G}$. The latter, of course, is

impossible in $\mathcal{D}'$. We completely describe the cases when the division is possible by obtaining a criterion of invertibility in $\mathcal{G}(\Omega)$.

The plan of our exposition is as follows. Section 2 presents some preliminaries. In Section 3 we extend the criterion of invertibility from the simplified version of Colombeau algebra $\mathcal{G}_s(\Omega)$ (see [4]) to its full version $\mathcal{G}(\Omega)$. The main result of the paper is given in Section 4, where we prove the global existence-uniqueness theorem within $\mathcal{G}(\Omega)$.

A novelty of the paper is that it treats singular coefficients in (1) in the context of mixed problems for a quite wide range of boundary conditions which can be classical as well as nonclassical (nonseparable and integral).

Existence-uniqueness results within Colombeau algebras for two-dimensional hyperbolic problems with discontinuous coefficients were studied in [6, 9, 11, 12]. Note that the discontinuity implies global boundedness estimates on the coefficients within Colombeau algebra $\mathcal{G}$, thereby avoiding the negative effect of infinite propagation speed. At the present paper we do not impose the assumption of global boundedness on coefficients of (1), thereby allowing them to be strongly singular. In [10] the authors use the Colombeau algebra of tempered generalized functions $\mathcal{G}_\tau$ to succeed with strongly singular coefficients in Cauchy problems for hyperbolic systems.

2. Preliminaries.

In this section we summarize the relevant material on Colombeau algebras of generalized functions.

Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. We denote by $\mathcal{G}(\Omega)$ and $\mathcal{G}(\overline{\Omega})$ the full version of Colombeau algebra of generalized functions over Ω and $\overline{\Omega}$, respectively. To define $\mathcal{G}(\Omega)$ and $\mathcal{G}(\overline{\Omega})$, we introduce the mollifier spaces in order to parametrize the regularizing sequences of generalized functions. For $q \in \mathbb{N}_0$ denote

$$\mathcal{A}_q(\mathbb{R}) = \Big\{ \varphi \in \mathcal{D}(\mathbb{R}) \;\Big|\; \int \varphi(x)\,dx = 1, \int x^k \varphi(x)\,dx = 0 \text{ for } \; 1 \le k \le q \Big\},$$

$$\mathcal{A}_q(\mathbb{R}^n) = \Big\{ \varphi(x_1, \ldots, x_n) = \prod_{i=1}^{n} \varphi_0(x_i) \;\Big|\; \varphi_0 \in \mathcal{A}_q(\mathbb{R}) \Big\}.$$

For $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$ define

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right).$$

We now introduce the algebra of moderate elements $\mathcal{E}_{\mathrm{M}}(\overline{\Omega})$ in the following way. Define

$$\mathcal{E}(\overline{\Omega}) = \{ u : \mathcal{A}_0 \times \overline{\Omega} \to \mathbb{R} \;\big|\; u(\varphi, .) \in C^\infty(\overline{\Omega}) \;\; \forall \varphi \in \mathcal{A}_0(\mathbb{R}) \}.$$

Now $\mathcal{E}_{\mathrm{M}}(\overline{\Omega})$, is defined to be a subalgebra of $\mathcal{E}(\overline{\Omega})$ consisting of elements $u \in \mathcal{E}(\overline{\Omega})$ with the following property:

$$\forall K \subset \overline{\Omega} \text{ compact } \;, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N} \text{ such that } \; \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n),$$

$$\exists C > 0, \exists \eta > 0 \text{ with } \; \sup_{x \in K} |\partial^\alpha u(\varphi_\varepsilon, x)| \le C \varepsilon^{-N}, \;\; 0 < \varepsilon < \eta.$$

The ideal $\mathcal{N}(\overline{\Omega})$ consists of all $u \in \mathcal{E}_{\mathrm{M}}(\overline{\Omega})$ such that

$$\forall K \subset \overline{\Omega} \text{ compact }, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N} \text{ such that } \forall q \geq N, \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n),$$

$$\exists C > 0, \exists \eta > 0 \text{ with } \sup_{x \in K} |\partial^\alpha u(\varphi_\varepsilon, x)| \leq C\varepsilon^{q-N}, \quad 0 < \varepsilon < \eta.$$

Finally,

$$\mathcal{G}(\overline{\Omega}) = \mathcal{E}_{\mathrm{M}}(\overline{\Omega})/\mathcal{N}(\overline{\Omega}).$$

This is an associative, commutative differential algebra. The algebra $\mathcal{G}(\Omega)$ on open set is constructed in the same manner (with Ω in place of $\overline{\Omega}$ in the definition above). Note that $\mathcal{G}(\Omega)$ admits a canonical embedding of $\mathcal{D}'(\Omega)$. We will use the notation $U = [(u(\varphi, x))_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)}]$ for elements U of Colombeau algebra $\mathcal{G}(\Omega)$ with $u(\varphi, x)$ to be a representative of U.

To reduce information from generalized functions to the level of distributions, we use the notion of an associated distribution. We say that $U \in \mathcal{G}(\Omega)$ admits $f \in \mathcal{D}'(\Omega)$ as associated distribution (or U is associated to f), denoted by $U \approx f$, if for all $\psi \in \mathcal{D}(\Omega)$ there exists $N \in \mathbb{N}$ such that

$$\lim_{\epsilon \to 0} \int u(\varphi_\varepsilon, x)\psi(x)\, dx = \langle f, \psi \rangle$$

for all $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$.

3. Criterion of invertibility in the full version of Colombeau algebra of generalized functions.

In spite of the fact that $\mathcal{G}(\Omega)$ is not a field, the division by singular distributions (in particular, by discontinuous functions and measures) is sometimes possible. It is given by the following criterion of (multiplicative) invertibility.

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary subdomain in $\mathbb{R}^n$.

THEOREM 1. *Let $U \in \mathcal{G}(\Omega)$ ($U \in \mathcal{G}(\overline{\Omega})$). Then the following two conditions are equivalent:*
(i) U is invertible in $\mathcal{G}(\Omega)$ (in $\mathcal{G}(\overline{\Omega})$), i.e., there exists $V \in \mathcal{G}(\Omega)$ ($V \in \mathcal{G}(\overline{\Omega})$) such that $UV = 1$ in $\mathcal{G}(\Omega)$ (in $\mathcal{G}(\overline{\Omega})$).
(ii) For each representative $(u(\varphi, x))_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)}$ of U and each compact set $K \subset \Omega$ ($K \subset \overline{\Omega}$) there exists $p \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$ there is $\eta > 0$ with $\inf_K |u(\varphi_\varepsilon, x)| \geq \varepsilon^p$ for all $0 < \varepsilon < \eta$.

Note that the criterion of invertibility for the simplified version of Colombeau algebra $\mathcal{G}_s(\Omega)$, where Ω is open, was proved in [4].

Proof. We use the argument similar to that presented in [4]. We prove the desired assertion for an arbitrary fixed open set Ω (the proof for the closed set $\overline{\Omega}$ is similar).

$(i) \Rightarrow (ii)$. Set $U = [(u(\varphi, x))_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)}]$ and $V = [(v(\varphi, x))_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)}]$. By assumption, there exists $N = [(n(\varphi, x))_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)}] \in \mathcal{N}(\Omega)$ such that $u(\varphi_\varepsilon, x)v(\varphi_\varepsilon, x) = 1 + n(\varphi_\varepsilon, x)$ for all $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$.

Fix an arbitrary compact set $K \subset \Omega$. We first prove that there exists $p \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$ there is $\eta > 0$ with $v(\varphi_\varepsilon, x) \neq 0$ for all $x \in K$ and $0 < \varepsilon < \eta$. Assume, to the contrary, that the latter is not true. This means that for each $p \in \mathbb{N}$ there exist $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$, a sequence $\varepsilon_n \searrow 0$, and a sequence $x_n \in K$ such that $v(\varphi_{\varepsilon_n}, x_n) = 0$ for all $n \geq 1$. Hence $0 = u(\varphi_{\varepsilon_n}, x_n)v(\varphi_{\varepsilon_n}, x_n) = 1 + n(\varphi_{\varepsilon_n}, x_n)$ and finally $n(\varphi_{\varepsilon_n}, x_n) = -1$ for all $n \geq 1$, a contradiction to the fact that $N \in \mathcal{N}(\Omega)$. Since $v \in \mathcal{E}_{\mathrm{M}}(\Omega)$, there exists $q \in \mathbb{N}$

such that for all $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ there are $C > 0$ and $\mu > 0$ with $\sup_K |v(\varphi_\varepsilon, x)| \leq C/\varepsilon^q$ for all $0 < \varepsilon < \mu$. Set $\tilde{q} = max\{p, q\}$. Due to the fact that $\mathcal{A}_{q+1}(\mathbb{R}^n) \subset \mathcal{A}_q(\mathbb{R}^n)$ for all $q \in \mathbb{N}_0$, we conclude that for each $\varphi \in \mathcal{A}_{\tilde{q}}(\mathbb{R}^n)$ the estimate

$$\inf_K |u(\varphi_\varepsilon, x)| \geq \frac{\varepsilon^q}{C}\Big(1 - \sup_K |n(\varphi_\varepsilon, x)|\Big) \geq \varepsilon^{q+1}$$

is true for all sufficiently small ε. Since K is an arbitrary compact subset of Ω, the desired assertion follows.

$(ii) \Rightarrow (i)$. Consider a covering $(K_i)_{i\in\mathbb{N}}$ of Ω by compact sets K_i such that $K_1 \subset K_2 \subset \cdots \subset \Omega$. It is known that, if $W \in \mathcal{G}(K_{i+1})$, then $W|_{K_j} \in \mathcal{G}(K_j)$ for all $j \leq i$. This fact is true due to the sheaf properties of $\mathcal{G}(\Omega)$.

Set $v(\varphi_\varepsilon, x) = 1/u(\varphi_\varepsilon, x)$ and $v_i(\varphi_\varepsilon, x) = v(\varphi_\varepsilon, x)|_{K_i}$. Fix an arbitrary $i \in \mathbb{N}$. By assumption, there exists $p \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$ there is a constant $\eta(\varphi) > 0$ such that the expression $1/u(\varphi_\varepsilon, x)$ exists for all $0 < \varepsilon < \eta(\varphi)$ and for all $x \in K_i$. For each $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$ let us set $v_i(\varphi_\varepsilon, x) \equiv 0$, where $0 < \varepsilon < \eta(\varphi)$ and $x \in K_i$. Consider the map $\varphi \to v_i(\varphi, x) : \mathcal{A}_0(\mathbb{R}^n) \to C^\infty(K_i)$. Let us show that this map is moderate. Indeed, for each $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$ we have

$$\sup_{K_i} |v_i(\varphi_\varepsilon, x)| = \frac{1}{\inf_{K_i} |u(\varphi_\varepsilon, x)|} \leq \frac{1}{\varepsilon^p}$$

for all sufficiently small $\varepsilon > 0$. The moderate estimate for $\partial^\alpha v_i(\varphi, x)$, where $|\alpha| = 1$, follows from the simple estimate

$$\left|\partial^\alpha\left(\frac{1}{u(\varphi_\varepsilon, x)}\right)\right| = \left|\frac{\partial^\alpha(u(\varphi_\varepsilon, x))}{u^2(\varphi_\varepsilon, x)}\right| \leq \frac{\partial^\alpha(u(\varphi_\varepsilon, x))}{\varepsilon^{2p}}$$

for sufficiently small ε and from the moderateness of $\partial^\alpha u(\varphi, x)$. Proceeding similarly with the higher-order derivatives of v_i, we conclude that $[(v_i(\varphi, x))_{\varphi\in\mathcal{A}_0(\mathbb{R}^n)}]$, denoted by V_i, belongs to $\mathcal{E}_{\mathrm{M}}(K_i)$. Furthermore, it is the inverse to U in $\mathcal{G}(K_i)$. From the definition of Colombeau generalized functions and the construction of V_i it follows that $V_i|_{K_j} \in \mathcal{G}(K_j)$ for all $j \leq i$. We therefore obtained a coherent family $\{V_i, i \in \mathbb{N}\}$. By the sheaf properties of $\mathcal{G}(\Omega)$, there exists a unique element $V \in \mathcal{G}(\Omega)$ such that $V|_{K_i} \in \mathcal{G}(K_i)$ for all $i \geq 1$. By construction, V is an inverse to U in $\mathcal{G}(\Omega)$. □

We now take into account the definition of Colombeau generalized numbers and the fact that an element $U \in \mathcal{G}(\Omega)$ is a constant iff there is $r \in \overline{\mathcal{C}}$ such that $U - r = 0$ in $\mathcal{G}(\Omega)$. The following corollary provides a criterion of invertibility of Colombeau generalized numbers within the full version of Colombeau algebras. For the same result within the simplified version of Colombeau algebras see [4].

COROLLARY 2. *Let $r \in \overline{\mathcal{C}}$. Then the following two conditions are equivalent:*
(i) r is invertible in $\overline{\mathcal{C}}$, i.e., there exists $s \in \overline{\mathcal{C}}$ with $rs = 1$ in $\overline{\mathcal{C}}$.
(ii) For each representative $(r(\varphi))_{\varphi\in\mathcal{A}_0(\mathbb{R})}$ of r there exists $p \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_p(\mathbb{R})$ there is $\eta > 0$ with $|r(\varphi_\varepsilon)| \geq \varepsilon^p$ for all $0 < \varepsilon < \eta$.

EXAMPLE 3. Let

$$U = \left[\left(l(\varphi) + \frac{1}{l^{m+1}(\varphi)}\Phi^{(m)}\left(\frac{x}{l(\varphi)}\right)\right)_{\varphi\in\mathcal{A}_0(\mathbb{R})}\right] \in \mathcal{G}(\Omega),$$

where $\Omega \subset \mathbb{R}$, $l(\varphi) = \sup\{|y|, \varphi(y) \neq 0\}$, $\Phi(x) \in \mathcal{D}(\Omega)$ is a fixed element of $\mathcal{D}(\Omega)$ such that $\int \Phi(x)\,dx = 1$ and $\Phi(x) \geq 0$. One can easily see that $U \approx \delta^{(m)}$. Indeed, for an arbitrary $\psi(x) \in \mathcal{D}(\Omega)$ we have

$$\lim_{\varepsilon \to 0} \int \left(\varepsilon l(\varphi) + \frac{1}{\varepsilon l^{m+1}(\varphi)} \Phi^{(m)} \left(\frac{x}{\varepsilon l(\varphi)} \right) \right) \psi(x)\,dx = < \delta^{(m)}, \psi > .$$

Since $l(\varphi_\varepsilon) = \varepsilon l(\varphi)$, we have the following estimate: for each compact set $K \subset \Omega$ and for each $\varphi \in \mathcal{A}_2(\mathbb{R})$ there exists $\eta > 0$ with

$$\inf_K \left| l(\varphi) + \frac{1}{l^{m+1}(\varphi)} \Phi^{(m)} \left(\frac{x}{l(\varphi)} \right) \right| = \inf_K \left| \varepsilon l(\varphi) + \frac{1}{\varepsilon l^{m+1}(\varphi)} \Phi^{(m)} \left(\frac{x}{\varepsilon l(\varphi)} \right) \right| \geq \varepsilon^2, \quad 0 < \varepsilon < \eta.$$

This estimate is uniform with respect to all compact sets $K \subset \Omega$ and $\varphi \in \mathcal{A}_2(\mathbb{R})$. By Theorem 1, U is invertible in $\mathcal{G}(\Omega)$.

This example shows that within $\mathcal{G}(\Omega)$ the division by the derivatives of the delta-function is possible.

PROPOSITION 4. *Let $U \in \mathcal{G}(\Omega)$ ($U \in \mathcal{G}(\overline{\Omega})$) and U is invertible in $\mathcal{G}(\Omega)$ (in $\mathcal{G}(\overline{\Omega})$). Then the multiplicative inverse of U is unique.*

Proof. We prove the desired assertion for an open set Ω (the proof for the closed set $\overline{\Omega}$ is similar).

Assume, to the contrary, that U possesses two multiplicative inverses $V_1, V_2 \in \mathcal{G}(\Omega)$. This implies the equality

$$U(V_1 - V_2) = 0 \text{ in } \mathcal{G}(\Omega).$$

We conclude from Theorem 1, specifically from the local invertibility estimate, that $U \notin \mathcal{N}(\Omega)$, hence that $V_1 - V_2 \in \mathcal{N}(\Omega)$, and finally that $V_1 = V_2$ in $\mathcal{G}(\Omega)$, a contradiction to our assumption. □

4. Existence and uniqueness of Colombeau generalized solutions.

In this section we develop the results of [8] to the case of singular coefficients in (1). Simultaniously, we consider less restrictive conditions on the initial data in (2) and (3). To prove a general global existence and uniqueness result in Colombeau algebra of generalized functions, we need the following definition of generalized functions of a less restrictive growth if comparing with $1/\varepsilon$-growth (see the definition of $\mathcal{E}_{\mathrm{M}}$).

DEFINITION 5. *Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. Suppose we have a function $\gamma : (0,1) \mapsto (0,\infty)$. An element $U \in \mathcal{G}(\Omega)$ ($U \in \mathcal{G}(\overline{\Omega})$) is called locally of γ-growth, if it has a representative $u \in \mathcal{E}_{\mathrm{M}}(\Omega)$ ($u \in \mathcal{E}_{\mathrm{M}}(\overline{\Omega})$) with the following property:*

For every compact subset $K \subset \Omega$ there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_{\mathcal{N}}(\mathbb{R}^n)$ there exist $C > 0$ and $\eta > 0$ with $\sup\limits_{x \in K} |u(\varphi_\varepsilon, x)| \leq C\gamma^N(\varepsilon)$ for $0 < \varepsilon < \eta$.

Note that this definition generalizes Definition 2 from [8].

DEFINITION 6. *Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. Suppose we have a function $\gamma : (0,1) \mapsto (0,\infty)$. An element $U \in \mathcal{G}(\Omega)$ ($U \in \mathcal{G}(\overline{\Omega})$) is called locally γ-invertible, if it has a representative $(u(\varphi, x))_{\varphi \in \mathcal{A}_0(\mathbb{R}^n)}$ with the following property:*

for each compact set $K \subset \Omega$ $(K \subset \overline{\Omega})$ *there exists* $p \in \mathbb{N}$ *such that for all* $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$ *there is* $\eta > 0$ *with* $\inf_K |u(\varphi_\varepsilon, x)| \geq \gamma^{-p}(\varepsilon)$ *for all* $0 < \varepsilon < \eta$.

We now make several assumptions on the initial data of problem (1)–(3). Let $\gamma(\varepsilon)$ and $\gamma_1(\varepsilon)$ be positive functions from $(0,1)$ to $(0,\infty)$ having the properties

$$\gamma(\varepsilon)^{\gamma^N(\varepsilon)} = O\left(\frac{1}{\varepsilon}\right), \quad \gamma_1(\varepsilon)^{\gamma_1^N(\varepsilon)} = O\left(\frac{1}{\varepsilon}\right), \quad \gamma(\varepsilon)^{\gamma_1^N(\varepsilon)} = O\left(\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0 \tag{4}$$

for each $N \in \mathbb{N}$. Assume that

1. The mapping $U \mapsto f(x,t,U)$ and all its derivatives are polynomially bounded, uniformly over (x,t) varying in compact subsets of $\overline{\Pi}$.
2. The mapping $U \mapsto \nabla_U f(x,t,U)$ is globally bounded, uniformly over (x,t) varying in compact subsets of $\overline{\Pi}$.
3. $\Lambda_1, \ldots, \Lambda_k < 0$, $\Lambda_{k+1}, \ldots, \Lambda_n > 0$ (these inequalities are satisfied on the level of representatives), where k is fixed and $1 \leq k \leq n$.
4. Λ_i and D_{ij} for $i \leq n$ and $j \leq n$ are locally of γ-growth on $\overline{\Pi}$.
5. B_{ij} and C_{ij} for $i \leq n$ and $j \leq n$ are locally of γ-growth on $[0,\infty)$.
6. $\partial_x \Lambda_i$ for $i \leq n$ are locally of γ_1-growth on $\overline{\Pi}$.
7. Λ_i for $i \leq n$ are locally γ-invertible on $\overline{\Pi}$.
8. The determinant of the matrix

$$R(t) = \begin{pmatrix} B_{1,k+1} & \ldots & B_{1n} & C_{11} & \ldots & C_{1k} \\ B_{2,k+1} & \ldots & B_{2n} & C_{21} & \ldots & C_{2k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{n,k+1} & \ldots & B_{nn} & C_{n1} & \ldots & C_{nk} \end{pmatrix}$$

   is locally γ-invertible on $[0,\infty)$.

9. $\operatorname{supp} A_i(x) \subset (-L, L)$; $\operatorname{supp} B_{ij}(t)$, $\operatorname{supp} C_{is}(t) \subset (0,\infty)$ for $1 \leq i \leq n$, $1 \leq j \leq k$, $k+1 \leq s \leq n$; $\operatorname{supp} D_{im}(x,t) \subset (0,\infty) \times [-L, L]$ for $1 \leq i, m \leq n$.

Let $U \in \mathcal{G}(\Omega)$ and a smooth function $g(x)$ be slowly increasing at the infinity. By the definition of $\mathcal{G}(\Omega)$, we have $g(U) \in \mathcal{G}(\Omega)$. Due to this fact and Assumption 1, $f(x,t,U)$ is a well-defined element of $\mathcal{G}(\Omega)$. Condition 2 is, in fact, sufficient and is imposed to ensure the global classical solvability of problem (1)–(3) with smooth initial data. We need Assumption 7 to transform the initial problem into an equivalent integral-operator form. Assumption 8 ensures the compatibility of (2) and (3) of any desired order. The hyperbolicity of system (1) is ensured by Assumption 3.

The point of Assumption 4 is that it allows us to consider Λ_i, $B_{ij}(t)$, $C_{ij}(t)$, and $D_{ij}(x,t)$ being discontinuous functions, the delta functions, and the derivatives thereof. An illustration of this fact is given by Example 3, if one takes $\gamma(l(\varphi))$ in place of $1/l(\varphi)$, where γ is specified

by $\gamma(p) = \sqrt{\frac{1}{2}\log\log\log(1/p)}$. If one takes in addition $\gamma_1(p) = \sqrt{\log\log(1/p)}$, the same example shows that Assumptions 4 and 5 on Λ do not contradict one another.

We are prepared to state the main result of the paper.

THEOREM 7 *Suppose that $A \in \mathcal{G}[-L, L]$, $\Lambda, D \in \mathcal{G}(\overline{\Pi})$, B, C, $H \in \mathcal{G}(\mathbb{R}_+)$, and f is smooth with respect to all its arguments. Under Assumptions 1–8 where the functions γ and γ_1 are specified by (4), problem (1)–(3) has a unique solution $U \in \mathcal{G}(\overline{\Pi})$.*

Proof. We first transform problem (1)–(3) into an equivalent integral-operator form. Note that all algebraic operations as well as operation of integration over finite intervals will be carried out on the level of representatives. Denote by $\omega_i(\tau; x, t)$ the i-th characteristic of (1) passing through a point $(x, t) \in \overline{\Pi}$, i.e., $\xi = \omega_i(\tau; x, t)$ is the solution to the Cauchy problem:

$$\frac{d\xi}{d\tau} = \Lambda_i(\xi(\tau), \tau), \quad \xi(t) = x.$$

The smallest value of $\tau \geq 0$ at which the characteristic $\xi = \omega_i(\tau; x, t)$ intersects $\partial\Pi$ will be denoted by $t_i(x, t)$.

By Assumption 7 and Theorem 1, $\det R(t)$ has an inverse with entries in $\mathcal{G}(\overline{\Pi})$. Using in addition Proposition 4, we conclude that there exists a unique element $(\det R)^{-1} \in \mathcal{G}(\overline{\Pi})$ such that $\det R\ (\det R)^{-1} = 1$. This means that the local part of boundary conditions (3) is solvable with respect to those components of U whose characteristics move into Π. Using this fact and integrating each equation of (1) along the corresponding characteristic curve, we obtain the following integral-operator form of (1)–(3):

$$\begin{aligned} U_i(x,t) = (R_i U)(x,t) + \int\limits_{t_i(x,t)}^{t} \Big[U(\omega_i(\tau;x,t),\tau) \int\limits_0^1 \nabla_U f_i(\omega_i(\tau;x,t),\tau,\sigma U)\, d\sigma \\ + f_i(\omega_i(\tau;x,t),\tau,0) \Big]\, d\tau, \quad 1 \leq i \leq n, \end{aligned} \tag{5}$$

where

$$(R_i U)(x,t) = \begin{cases} M_i(t_i(x,t)) \text{ if } t_i(x,t) > 0, \\ \\ A_i(\omega_i(0;x,t)) \text{ if } t_i(x,t) = 0, \end{cases}$$

$$M_i(t) = U_i|_{x=-L}, \quad k+1 \leq i \leq n$$
$$M_i(t) = U_i|_{x=L}, \quad 1 \leq i \leq k,$$

and

$$\begin{aligned} M_i(t) = \frac{1}{\det R(t)} \sum_{j=1}^{n} R_{ji}^{ad}(t) \Big[H_j(t) - \sum_{s=1}^{k} B_{js}(t) U_s(-L,t) \\ - \sum_{s=k+1}^{n} C_{js}(t) U_s(L,t) - \sum_{s=1}^{n} \int\limits_{-L}^{L} D_{js}(x,t) U_s(x,t)\, dx \Big]. \end{aligned}$$

It is easy to see that problems (1)–(3) and (5) are equivalent in $\mathcal{G}(\Omega)$.

Given $T > 0$, denote

$$\Pi^T = \{(x,t) | -L < x < L, 0 < t < T\}.$$

In [8] we proved that problem (1)–(3) with smooth initial data has a unique smooth solution in Π^T, whatsoever $T > 0$. For this purpose we used the contraction mapping principle and obtained local smooth solution. In parallel, we obtained local a priori estimates for the latter. To obtain global smooth solution, we used finite iteration of the local a priori estimates. We also derived global a priori estimates for this solution. To prove the existence of a generalized solution to the problem under consideration, let us rewrite just mentioned estimates from [8], with taking care of the norm of Λ_i as well as of the norms of the elements of R. Notice that the assumptions imposed on Λ_i and R here differ from those imposed in [8]. To be precise, in [8] Λ_i and R_{ij} for all $i, j \leq n$ are assumed to be, respectively, smooth and Colombeau generalized functions locally of bounded growth. Referring the reader to [8] for details, we now write down the final a priori estimates for a global smooth solution U in a suitable for our purposes form. Set

$$E_U(l) = \max\{|\partial_x^l U_i(x,t)| : (x,t) \in \overline{\Pi}^T, 1 \leq i \leq n\},$$

$$E_{\Lambda,max}(l_1,l_2) = \max\{|\partial_x^{l_1}\partial_t^{l_2}\Lambda_i(x,t)| : (x,t) \in \overline{\Pi}^T, 1 \leq i \leq n\},$$

$$E_{\Lambda,min} = \min\{|\Lambda_i(x,t)| : (x,t) \in \overline{\Pi}^T, 1 \leq i \leq n\},$$

$$E_R = \max_{t\in[0,T]}\left|\frac{1}{\det R(t)}\right|,$$

$$E_B(l) = \max\{|B_{ij}^{(l)}(t)| : t \in [0,T], 1 \leq i,j \leq n\},$$

$$E_D(l) = \max\{|\partial_t^l D_{ij}(x,t)| : (x,t) \in \overline{\Pi}^T, 1 \leq i,j \leq n\},$$

$$E_F = \max\{|\nabla_U f_i(x,t,y)| \,:\, (x,t,y) \in \overline{\Pi}^T \times \mathbb{R}, 1 \leq i \leq n\}$$

$$q_0 = n^2 \max_{\substack{t\in[0,T],\\ 1\leq i,j\leq n}}\left|\frac{R_{ji}^{ad}(t)}{R(t)}\right|\Bigg[nE_F\Bigg(\max_{\substack{t\in[0,T],1\leq j\leq n,\\ 1\leq s\leq k,k+1\leq r\leq n}}\{|B_{js}(t)|,|C_{jr}(t)|\} + 2LE_D(0)|\Bigg)$$
$$+E_D(0)E_{\Lambda,max}(0,0)\Bigg] + nE_F,$$

$$q_m = (q_0 - nE_F)E_{\Lambda,max}(0,0)^m(E_{\Lambda,min})^{-m} + nE_F + mE_{\Lambda,max}(1,0).$$

With this notation, we have

$$E_U(m) \leq P_{1,m}\Bigg(\frac{1}{1-q_m t(m)}, \max\{E_B^n(0), E_C^n(0)\}, E_R, E_D(0), n^2, L,$$
$$(E_{\Lambda,max}(0,0))^m, (E_{\Lambda,min})^{-m}\Bigg)$$
$$\times P_{2,m}\Bigg(\max_{0\leq l\leq m-1} E_U(l), (E_{\Lambda,min})^{-\mathrm{sgn}(m)}, \max_{1\leq l_1+l_2\leq m} E_{\Lambda,max}(l_1,l_2), \max_{0\leq l\leq m}\{E_B(l), E_C(l)\}, \tag{6}$$
$$\max_{0\leq l\leq m} E_D(l), E_R, \max_{\substack{t\in[0,T],\\ 1\leq i\leq n,0\leq l\leq m}} |H_i^{(l)}(t)|, \max_{\substack{x\in[-L,L],\\ 1\leq i\leq n,0\leq l\leq m}} |A_i^{(l)}(x)|\Bigg),$$

where $t(m) \le \min\{L/E_{\Lambda,max}(0,0), 1/q_m\}$, $P_{1,m}$ is a polynomial of degree $8\lceil T/t(m)\rceil$ with positive constant coefficients not depending on ε, and $P_{2,m}$ is a polynomial whose degree depends on m but neither on T nor on $t(m)$ (and, therefore, not depending on ε) with positive constant coefficients depending on f and not depending on ε.

The classical smooth solution to problem (1)–(3) satisfying estimates (6) in Π^T for all $m \in \mathbb{N}_0$ can be constructed by the sequential approximation method. This solution will serve to build up a representative of the Colombeau solution. We now construct the latter. Accordingly to the assumptions of the theorem, we consider all the initial data as elements of the corresponding Colombeau algebras. We choose representatives λ, a, b, c, d, and h of Λ, A, B, C, D, and H, respectively, with the properties required in the theorem. Hence a representative of R is therewith defined. We will denote it by r. Let $\phi = \varphi \otimes \varphi \in \mathcal{A}_0(\mathbb{R}^2)$. Consider a prospective representative $u = u(\phi, x, t)$ of U which is the classical smooth solution to problem (1)–(3) with initial data $\lambda(\phi, x, t)$, $a(\varphi, x)$ and boundary data $b(\varphi, t)$, $c(\varphi, t)$, $d(\phi, x, t)$, $h(\varphi, t)$. It remains to show the moderateness of u, i.e. that $u \in \mathcal{E}_{\mathrm{M}}$. To do so, we will obtain moderate growth estimates of $u(\phi_\varepsilon, x, t)$ in terms of the regularization parameter ε.

Let ε be small enough and $\phi \in \mathcal{A}_N(\mathbb{R}^2)$ with N chosen so large that the following conditions are true:

a) the moderation property holds for $a(\varphi_\varepsilon, x)$ and $h(\varphi_\varepsilon, t)$;

b) the local-γ-invertibility estimate (see Definition 6) holds for $\lambda_i(\phi_\varepsilon, x, t)$ and $r(\varphi_\varepsilon, t)$.

c) the local-γ-growth estimate (see Definition 5) holds for $\lambda_i(\phi_\varepsilon, x, t)$, $b_{ij}(\varphi_\varepsilon, t)$, $c_{ij}(\varphi_\varepsilon, t)$ and $d_{ij}(\phi_\varepsilon, x, t)$, where $i \le n$ and $j \le n$.

d) the local-γ_1-growth estimate holds for $\partial_x \lambda_i(\phi_\varepsilon, x, t)$, where $i \le n$.

It suffices to prove the moderateness of $P_{1,m}$ and $P_{2,m}$ for all $m \in \mathbb{N}_0$, where $U(x,t)$, $\Lambda(x,t)$, $A(x)$, $B(t)$, $C(t)$, $R(t)$, $D(x,t)$, and $H(t)$ are replaced by their representatives $u(\phi, x, t)$, $\lambda(\phi, x, t)$, $a(\varphi, x)$, $b(\varphi, t)$, $c(\varphi, t)$, $r(\varphi, t)$, $d(\phi, x, t)$, and $h(\varphi, t)$, respectively. We see at once that for each $m \in \mathbb{N}_0$ the estimate

$$q_m \le \gamma^{2N(m+1)+1}(\varepsilon) + \gamma_1^{N+1}(\varepsilon)$$

is true for all sufficiently small ε. Since $t(m) \le \min\{L/E_{\Lambda,max}(0,0), 1/q_m\}$ and $E_{\Lambda,max}(0,0) \ge 1/\gamma^N(\varepsilon)$ for all $\varphi \in \mathcal{A}_N(\mathbb{R})$, we can choose $t(0) = 1/[2(\gamma^{2N(m+1)+1}(\varepsilon) + \gamma_1^{N+1}(\varepsilon))] < 1/q_0$. Taking into account (4), for each $m \in \mathbb{N}_0$ and for all small enough ε we have

$$\left(\frac{1}{1 - q_m t(m)}\right)^{\lceil T/t(m)\rceil} \le 2^{\lceil 2T(\gamma^{2N(m+1)+1}(\varepsilon) + \gamma_1^{N+1}(\varepsilon))\rceil}$$

$$\le \left(\gamma(\varepsilon)^{\gamma^{2N(m+1)+1}(\varepsilon)}\right)^{\lceil 2T\rceil+1} \left(\gamma_1(\varepsilon)^{\gamma_1^{N+1}(\varepsilon)}\right)^{\lceil 2T\rceil+1} = O\left(\frac{1}{\varepsilon}\right),$$

$$\left(\max\{E_b^n(0), E_c^n(0)\} E_r E_d(0) n^2 L (E_{\lambda,max}(0,0))^m (E_{\lambda,min}(0))^{-m}\right)^{\lceil T/t(m)\rceil}$$

$$\le \gamma(\varepsilon)^{N(2m+n+2)\lceil 2T(\gamma^{2N(m+1)+1}(\varepsilon) + \gamma_1^{N+1}(\varepsilon))\rceil} = O\left(\frac{1}{\varepsilon}\right)$$

It follows that for each $m \in \mathbb{N}_0$ there exists $N \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_N(\mathbb{R})$ we have

$$P_{1,m}\Big(\frac{1}{1-q_m t(m)}, \max\{E_b^n(0), E_c^n(0)\}, E_r, E_d(0)n^2, L, \\ (E_{\lambda,max}(0,0))^m, (E_{\lambda,min})^{-m}\Big) = O\Big(\frac{1}{\varepsilon}\Big). \tag{7}$$

One can easily see now that for $l = 0$

$$E_u(l) = O\Big(\frac{1}{\varepsilon}\Big) \tag{8}$$

for all $\varphi \in \mathcal{A}_N(\mathbb{R})$ with large enough $N \in \mathbb{N}$. To prove similar estimates for all derivatives of U_i with respect to x, we use induction on l. Assuming (8) to hold for $l \leq m$, let us show that (8) is true for $l = m+1$ as well. Indeed, let ε be small enough and $\varphi \in \mathcal{A}_N(\mathbb{R})$ with N chosen so large that the following conditions are true:

a) the moderateness property holds for $\partial^s a(\varphi_\varepsilon, x)$, $\partial^s h(\varphi_\varepsilon, t)$, $b^{(s)}(\varphi_\varepsilon, t)$, $c^{(s)}(\varphi_\varepsilon, t)$, $\partial_t^s d(\phi_\varepsilon, x, t)$, $\partial_x^{l_1}\partial_t^{l_2}\lambda(\phi_\varepsilon, x, t)$, $\partial_x^l u(\phi_\varepsilon, x, t)$ for all $0 \leq s \leq m+1$, $0 \leq l \leq m$, $0 \leq l_1+l_2 \leq m+1$;

b) the local-γ-invertibility estimate holds for $\lambda_i(\phi_\varepsilon, x, t)$.

Note that $\partial_x^l u(\phi_\varepsilon, x, t)$ for $0 \leq l \leq m$ has moderateness property due to the induction assumption. Since $P_{2,m}$ is a polynomial whose degree does not depend on ε, the moderateness of $P_{2,m}$ becomes obvious. We are done by (7).

The moderate estimates on t as well as on mixed derivatives follow immediately from (1) by successive differentiation. This finishes the existence part of the proof.

The proof of the uniqueness part follows the same scheme. The only difference is that now we consider problem (1)–(3) with right hand sides of (2) and (3) in $\mathcal{N}$. The analysis is even simplier since by [2], it is sufficient to check negligibility at order zero. The proof is complete. □

REMARK 8. To prove the theorem, we used an integral-operator form (5) of the problem under consideration. Considering (5) with respect to a Colombeau function $U \in \mathcal{G}(\overline{\Pi}^T)$, we see that the right hand side of (5) includes compositions of generalized functions. Specifically, we have compositions of the singular initial and boundary data as well as the function U with the singular characteristic curves.

Note that the Colombeau algebra $\mathcal{G}$ is invariant under superposition with smooth polynomially bounded maps. In spite of the fact that the latter is not the case for the compositions involved by (5), all terms in (5) are well-defined in the Colombeau sense. To show this, consider system (5) with U replaced by $u(\phi, x, t)$, where the latter is a representative of the Colombeau solution stated in the theorem. From the proof it follows that, given $(x,t) \in \overline{\Pi}^T$, the domain of dependence for $u(\phi_\varepsilon, x, t)$ is included in a compact subset of $\overline{\Pi}^T$ which is independent of $\varepsilon > 0$ and $\varphi \in \mathcal{A}_0(\mathbb{R})$. This means that we here do not have the effect of infinite propagation speed (which could be caused by the fact that characteristic curves depend on $\varphi \in \mathcal{A}_0(\mathbb{R})$).

We conclude that the right hand side of (5) is well-defined in the Colombeau sense.

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